

Using a Single Extra Constraint to Linearize the Quadratic Assignment Problem

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ABSTRACT

The paper presents a new powerful technique to linearize the quadratic assignment problem. There are so many techniques available in literature that are used to linearize the quadratic assignment problem. In all these linear formulations, both the number of variables and linear constraints significantly increase. The technique proposed in this paper has the strength that the number of linear constraints increases by only one after linearization process. The QAP has application in areas such as wring, hospital layout, dartboard design, typewriter keyboard design, production process, and scheduling.

KEYWORDS

Koopmans and Beckmann Formulation, Linear Binary Form, Quadratic Assignment Problem

1. INTRODUCTION

The quadratic assignment problem (QAP) is a well-known problem and this is a problem whereby a set of facilities are allocated to a set of locations in such a way that the cost is a function of the distance and flow between the facilities. In this problem the costs are associated with a facility being placed at a certain location. The objective is to minimize the assignment of each facility to a location as given Munapo (2012) and Koopmans & Beckmann (1957). There are three main categories of methods for solving the quadratic assignment problem. These categories are heuristics, bounding techniques and exact algorithms.

1.1 Heuristics

These are algorithms that quickly give near optimal solutions to the quadratic assignment problem that are given in Drezner (20008) and Yang et al. (2008). There are five main classes of heuristics for the quadratic assignment problem and these are:

1. Construction methods
2. Limited enumeration methods
3. Improvement methods
4. Simulated annealing techniques
5. Genetic algorithms

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1.2 Bounding Techniques

For a formulated quadratic assignment problem a lower bound can be calculated. There are several types of bounds that can be calculated for a quadratic assignment problem as given in Adams & Johnson (1994) and Ramakrishnan (2002). These are:

1. Gilmore-Lawler bounds
2. Eigenvalue related bounds
3. Bounds based on reformulations

Lower bounds are important in two main ways. Besides being used to approximate optimal solutions they can be used within the context of heuristics or exact methods.

1.3 Exact Algorithms

There are four main classes of methods for solving the quadratic assignment problem exactly as given in Cela (1998) and Nagarajan and Sviridenko (2009). These are:

1. Dynamic programming
2. Cutting plane techniques
3. Branch and bound procedures
4. Hybrids of the last two

Research on these four methods has shown that the hybrids are most successful for solving instances of the quadratic assignment problem.

1.4 Applications of the QAP

The QAP has application in wiring, hospital layout, dartboard design, typewriter keyboard design, production process, scheduling etc.

1.4.1 Steinberg Wiring Problem

When wiring a computer backboard, there is a need to minimize the total amount or length of wire used. The main reason we need to minimize the amount of wire or length of wire is to minimize costs. In addition minimizing the total length of the wiring will improve computing time. To achieve this the wiring problem is formulated as a QAP and this problem is now known as Steinberg Wiring Problem.

1.4.2 Hospital Layout

In designing a hospital layout there are so many important factors that must be considered. These important factors include the patients, hospital staff, clinics, X-ray room, emergency room, drug store etc. In designing the hospital layout the objective is to minimize the total distance a patient in need of urgent care must travel before being treated. This problem is formulated as a QAP.

1.4.3 Dartboard Design

A competitive sport in which two or more players bare-handedly throw small sharp-pointed missiles at a round target or dartboard is called darts or dart-throwing. In darts, points are scored by hitting specific marked areas of the board. These areas follow a principle of points increasing towards the centre of the board. The dartboard design problem can be formulated as a QAP.

1.4.4 Typewriter Keyboard Design

The use of smart phone and tablet is increasing significantly these days.

For one to enter data or text on these modern devices, virtual keyboards are now being used instead of the conventional hardware keyboards. The challenge is what is the best virtual keyboard layout for these devices? This problem is modeled as a quadratic assignment problem.

1.4.5 Production

In production processes orders for a number of products must be scheduled on a number of similar production lines so as to minimize the sum of product- dependent changeover costs, production costs and time-constraint penalties. This is a production problem that can be modeled as a quadratic assignment problem.

1.4.6 Scheduling

Scheduling is very important in big hospitals, large universities, rail operations, large bus companies, airlines, etc. As an example, assignment of classes at a university can be scheduled in such a way that very few similar classes would be in the same time slot. In order to do this, the problem can be formulated as a quadratic assignment problem.

More on applications of the quadratic assignment problem can be found in Mohamed et al. (2018).

2. QUADRATIC ASSIGNMENT PROBLEM FORMULATION

2.1 Types Quadratic Assignment Problems

There are four main variants of the quadratic assignment problem. These are the quadratic bottleneck assignment problem (QBAP), the biquadratic assignment problem (BQAP), the quadratic semi-assignment problem (QSAP), and the generalized quadratic assignment problem (GQAP).

2.1.1 Quadratic Bottleneck Assignment Problem (QBAP)

Suppose we are given a set of n facilities and a set of n locations. Suppose it is also given that for each pair of locations, a *distance* is specified and for each pair of facilities a *weight* or *flow* is also specified. The quadratic bottleneck problem is to the problem of assigning all facilities to different locations with the goal of minimizing the maximum of the distances multiplied by the corresponding flows.

2.1.2 BiQuadratic Assignment Problem (BQAP)

A biquadratic assignment problem can be defined as a quartic assignment problem with cost coefficients formed by the products of two four-dimensional arrays.

2.1.3 Quadratic Semi-Assignment Problem (QSAP)

In the quadratic semi-assignment problem (QSAP) we are given again two coefficient matrices, which are the flow matrix and a distance matrix. In this case there are n objects and m locations and are in such a way that $n > m$. The objective is to assign all objects to locations and at least one object to each location so as to minimize the overall distance covered by the flow of materials (or people) moving between different objects.

2.1.4 Generalized Quadratic Assignment Problem (GQAP)

The GQAP involves the minimization of a total pair-wise interaction cost among m equipment, tasks or other entities and where placement of these entities into n possible destinations and is dependent upon existing resource capacities.

For more on quadratic assignment variants readers are encouraged to see Billionnet & Elloumi (2001). There are so many mathematical formulations for QAP. In this chapter we use the linear form proposed by Munapo (2012). This linear form is an extension of the formulation introduced

by Koopmans and Beckmann (1957). In this formulation we assume that new buildings are to be placed on a piece of land and n sites have been identified as sites for the buildings. We also assume that each building has a special function.

2.2 Koopmans-Beckmann Formulation

Let:

- a_{ij} be the walking distance between sites i and j .
- b_{kl} be the number of people per week who circulate between buildings k and l .

Then the Koopmans- Beckmann formulation of the QAP is given as:

$$\text{Maximize } Z = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n a_{ij} b_{kl} x_{ik} x_{jl} + \sum_{i=1}^n \sum_{k=1}^n c_{ik} x_{ik}$$

such that:

$$\sum_{i=1}^n x_{ij} = 1, \quad 1 \leq j \leq n \quad (1)$$

$$\sum_{j=1}^n x_{ij} = 1, \quad 1 \leq i \leq n$$

$$x_{ij} \in \{0,1\}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n$$

In this formulation there are n^2 variables and $2n$ constraints (Koopmans & Beckman, 1957).

3. CURRENT LINEARIZATION TECHNIQUE

The current technique can linearize the Koopmans-Beckmann model to the form given in (2):

$$\text{Maximize } z = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n a_{ij} b_{kl} y_{ijkl},$$

such that:

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n y_{ijkl} = n^2 \quad (2)$$

$$x_{ik} + x_{jl} \geq 2y_{ijkl}, \quad \forall i, j, k, l$$

$$y_{ij} \geq x_{ik} + x_{jl} - 1$$

Solving this linearized QAP model becomes very difficult as n increases in size. This linearized model has $(n^4 + n^2)$ variables and $O(n^4)$ constraints. This is very difficult to manage as n becomes large.

3.1 The General Quadratic Binary Problem

The Koopmans-Beckmann is a special case of a quadratic binary problem.

Let a general case of the quadratic binary problem be represented in (3):

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij}^0 x_i x_j + \sum_k c_k^1 x_k$$

such that:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \quad (3)$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

where a_{ij}, b_i, c_{ij}^0 and c_k^1 are constants:

$$1 \leq i \leq m, \quad 1 \leq j \leq n$$

$$x_i, x_j, x_k \in \{0,1\}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n, \quad 1 \leq k \leq n$$

3.2 The Variables $x_i x_j$ Where $i = j$

If $i = j$ then $x_i^2 = x_j^2$. For binary integer variables we have the following:

$$x_i(x_i - 1) = 0$$

$$x_i^2 - x_i = 0$$

$$x_i = x_i^2 \quad (4)$$

Thus x_i^2 can be replaced by x_i in the objective function. Similarly x_j^2 can also be replaced by x_j in the objective function. Note that this substitution on its own does not change the number of variables in the problem.

3.3 The Variables $x_i x_j$ Where $i \neq j$

If $i \neq j$ then in the worst case there are $\frac{n(n-1)}{2}$ combinations of such variables in the objective function.

Proof 1: Suppose

- Two variables (x_1 and x_2): Then in the worst case we can have $x_1 x_2$ as the only possible combination of variables.
- Three variables (x_1, x_2 and x_3): Then in the worst case we can have $x_1 x_2, x_1 x_3$ or $x_2 x_3$ as the possible combinations of variables. Thus these three variables give 3 possible combinations.

◦ n variables ($x_1, x_2, \dots, x_{n-1}$ and x_n): Then in the worst case we can have:

$$x_1x_2, x_1x_3, \dots, x_1x_n, x_2x_3, x_2x_4, \dots, x_2x_n, \dots, x_{n-1}x_n$$

as the possible combinations. This results in:

$$(n-1) + (n-2) + \dots + 1 = \sum_{t=1}^{n-1} t = \frac{n(n-1)}{2}$$

possible combinations.

3.4 Linearizing the Quadratic Binary Problem

The variable combinations $x_i x_j$ where $i \neq j$ must be removed in order to make the objective function linear. This is done by using the following substitution.

3.4.1 Variable Substitution

Let:

$$x_i x_j = \delta_r \tag{5}$$

where δ_r is also a binary variable and $r = 1, 2, \dots, \frac{n(n-1)}{2}$, such that:

$$x_i + x_j = 2\delta_r + \bar{\delta}_r$$

$$\delta_r + \bar{\delta}_r \leq 1 \tag{6}$$

$$\delta_r, \bar{\delta}_r \in \{0, 1\} \text{ and } r = 1, 2, \dots, \frac{n(n-1)}{2}$$

3.4.2 Proof 2

In this case we have to show that the solution space $\Omega(x_i x_j) = \{0, 1\}$ is also the solution space for $\Omega(\delta_r)$, every point in $\Omega(x_i x_j)$ has a corresponding point in $\Omega(\delta_r)$ and that $x_i x_j = \delta_r$ for all corresponding points.

3.4.2.1 Solution Space for $x_i x_j$ i.e. $\Omega(x_i x_j)$

$$\left. \begin{array}{l} x_i = 0 \text{ and } x_j = 0, \\ x_i = 1 \text{ and } x_j = 0, \\ x_i = 0 \text{ and } x_j = 1, \end{array} \right\} x_i x_j = 0$$

$$x_i = 1 \text{ and } x_j = 1 \Rightarrow x_i x_j = 1$$

$$\therefore \Omega(x_i x_j) = \{0, 1\}$$

3.4.2.2 Solution Space for δ_r i.e. $\Omega(\delta_r)$

$$\delta_r = 1 \text{ and } \bar{\delta}_r = 0 \Rightarrow x_i + x_j = 2 \Rightarrow x_i = x_j = 1 \Rightarrow x_i x_j = 1$$

$$\delta_r = 0 \text{ and } \bar{\delta}_r = 1 \Rightarrow x_i + x_j = 1, \Rightarrow \begin{cases} \text{either } x_i = 1 \text{ and } x_j = 0 \Rightarrow x_i x_j = 0 \\ \text{or } x_i = 0 \text{ and } x_j = 1 \Rightarrow x_i x_j = 0 \end{cases}$$

$$\delta_r = 0 \text{ and } \bar{\delta}_r = 0 \Rightarrow x_i + x_j = 0 \Rightarrow x_i = x_j = 0 \Rightarrow x_i x_j = 0$$

$$\therefore \Omega(\delta_r) = \{0, 1\}$$

3.4.2.3 Corresponding Points

Point in $\Omega(x_i x_j)$. Corresponding point in $\Omega(\delta_r)$:

$$x_i = 0 \text{ and } x_j = 0, \delta_r = 0 \text{ and } \bar{\delta}_r = 0$$

$$x_i = 1 \text{ and } x_j = 0, \delta_r = 0 \text{ and } \bar{\delta}_r = 1$$

$$x_i = 0 \text{ and } x_j = 1, \delta_r = 0 \text{ and } \bar{\delta}_r = 1$$

$$x_i = 1 \text{ and } x_j = 1, \delta_r = 1 \text{ and } \bar{\delta}_r = 0$$

3.5 Number of Variables and Constraints in the Linearized Model

In the linearization process two extra variables are added to every product of variables $x_i x_j$ where $i \neq j$, that appears in the objective function. In other words for any quadratic binary problem there are $\frac{n(n-1)}{2}$ such products as shown in Section 2. Thus there are:

$$2 \times \frac{n}{2}(n-1) = n(n-1) \text{ new variables} \quad (7)$$

This gives a total of:

$$n(n-1) \text{ new variables} + n \text{ original variables} = n^2 \text{ variables}$$

Also two extra constraints are added for every product of variables $x_i x_j$ where $i \neq j$ that appears in the objective function. The total number of new constraints is given by (8):

$$2 \times \frac{n}{2}(n-1) = n(n-1) \text{ constraints} \quad (8)$$

The total number of constraints ($\bar{m}$) is given by (9):

$$\bar{m} = m \text{ original constraints} + n(n-1) \text{ original constraints} = (n^2 + m - n) \text{ variables} \quad (9)$$

3.6 Linearized Quadratic Binary Problem

Then linearized model becomes as given in (10):

$$\text{Minimize } Z = \sum_{r=1}^{\frac{n(n-1)}{2}} \bar{c}_r^0 \delta_r + \sum_k^n \bar{c}_k^1 x_k,$$

such that:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \quad (10)$$

$$x_i + x_j = 2\delta_r + \bar{\delta}_r, \forall i \neq j$$

$$\delta_r + \bar{\delta}_r \leq 1, \forall i \neq j$$

$$x_i, x_j, x_k \in \{0,1\}, 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n$$

$$\delta_r, \bar{\delta}_r \in \{0,1\} \text{ and } r = 1, 2, \dots, \frac{n(n-1)}{2}$$

3.7 Reducing the Number of Extra Constraints in the Linear Model

Solving a linear model with $n(n-1)$ extra constraints and $(n^2 + m - n)$ extra variables becomes very difficult for large QAPs. It is possible to halve the number of extra constraints. The following constraints can be combined into one:

$$x_i + x_j = 2\delta_r + \bar{\delta}_r$$

$$\delta_r + \bar{\delta}_r \leq 1$$

The first constraint can be expressed as given in (11) and (12):

$$x_i + x_j = \delta_r + \delta_r + \bar{\delta}_r \quad (11)$$

$$x_i + x_j - \delta_r = \delta_r + \bar{\delta}_r \quad (12)$$

Since $\delta_r + \bar{\delta}_r$ cannot exceed one as given in (13):

$$x_i + x_j - \delta_r \leq 1 \quad (13)$$

This reduces the number of extra constraints and variables to $\frac{n(n-1)}{2}$.

3.8 Further Reduction of Extra Constraints

The reduced $\frac{n(n-1)}{2}$ extra constraints still poses a serious problem as n increases. For example if $n = 1000$ it means both constraints and variables increase by $\frac{1000(1000-1)}{2} = 499\ 500$. There is a need to reduce these number of constraints to lowest number possible. The lowest possible number of extra constraints is one and can be done. The $\frac{n(n-1)}{2}$ extra constraints $x_i + x_j - \delta_r \leq 1$ can be combined into one surrogate constraint:

$$\begin{aligned} x_1 + x_2 - \delta_1 &\leq 1 \\ x_1 + x_3 - \delta_2 &\leq 1 \\ &\dots \\ x_{\ell-1} + x_{\ell} - \delta_{\ell} &\leq 1 \end{aligned} \quad (14)$$

where:

$$\ell = \frac{n(n-1)}{2}$$

Combining the constraints we have (16):

$$x_1 + x_2 - \delta_1 + x_1 + x_3 - \delta_2 + \dots + x_{\ell-1} + x_{\ell} - \delta_{\ell} \leq 1 + 1 + \dots + 1 \quad (15)$$

$$(n-1)(x_1 + x_2 + \dots + x_n) - (\delta_1 + \delta_2 + \dots + \delta_{\ell}) \leq \ell \quad (16)$$

Then linearized model becomes as given in (17):

$$\text{Minimize } Z = \sum_{r=1}^{\frac{n(n-1)}{2}} \bar{c}_r^0 \delta_r + \sum_i^n \bar{c}_k^1 x_k$$

such that:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\dots \end{aligned} \quad (17)$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$(n-1)(x_1 + x_2 + \dots + x_n) - (\delta_1 + \delta_2 + \dots + \delta_\ell) \leq \ell$$

Note that the number of constraints increases by only one as given in (17).

4. NUMERICAL ILLUSTRATION

Solve the quadratic assignment problem given in (18):

$$\text{Minimize } z = 25x_1 + 21x_2 + 22x_3 + 19x_4 + 14x_1x_2 + 15x_1x_3 + 17x_1x_4 \\ + 17x_2x_3 + 21x_2x_4 + 20x_3x_4$$

such that:

$$13x_1 + 21x_2 + 19x_3 + 20x_4 \geq 52 \tag{18}$$

$$23x_1 + 17x_2 + 21x_3 + 26x_4 \geq 57$$

$$20x_1 + 24x_2 + 17x_3 + 22x_4 \geq 54$$

$$x_j \in \{0,1\}, j = 1,2,3,4$$

4.1 Making the Model Linear

Using $x_j = x_j^2$ and $x_i x_j = \delta_r$, the linear model in the numerical illustration becomes as given in (19):

$$\text{Minimize } z = 47x_1 + 45x_2 + 45x_3 + 40x_4 + 14\delta_1 + 15\delta_2 + 17\delta_3 + 17\delta_4 + 21\delta_5 + 20\delta_6$$

such that:

$$13x_1 + 21x_2 + 19x_3 + 20x_4 \geq 52$$

$$23x_1 + 17x_2 + 21x_3 + 26x_4 \geq 57 \tag{19}$$

$$20x_1 + 24x_2 + 17x_3 + 22x_4 \geq 54$$

$$3x_1 + 3x_2 + 3x_3 + 3x_4 - \delta_1 - \delta_2 - \delta_3 - \delta_4 - \delta_5 - \delta_6 \leq 6$$

where:

$$\delta_i \in \{0,1\}, i = 1,2,3,4,5,6$$

4.2 Solving as a Linear Integer Model

Solving as a linear model, the optimal solution is obtained as given in (20):

$$\begin{aligned} x_2 = x_3 = x_4 = \delta_1 = \delta_2 = \delta_3 &= 1, \\ x_1 = \delta_4 = \delta_5 = \delta_6 &= 0 \end{aligned} \tag{20}$$

5. CONCLUSION

One of the reasons why the quadratic assignment problem appears to be difficult, is because of the huge numbers of extra constraints and variables that result after the linearization process. In the proposed approach the number of extra constraints is one. The resulting linear integer problem after the linearization process is a zero-one problem and this problem can be solved efficiently by interior point algorithms as given in Munapo (2016 & 2020). The quadratic assignment problem presented in this paper is not as difficult as was believed to be.

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