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Sheaf Representation of an Information System

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ABSTRACT

Ever since Pawlak introduced the concepts of rough sets, it has attracted many researchers and scientists from various fields of science and technology. Particularly for algebraists as it presented a gold mine to explore the algebraic and topological connections with rough set theory. The present article deals with the connections between rough sets and sheaves. The authors studied sheaf representation of an information system in rough set framework and illustrated how it helps information retrieval.

KEYWORDS

Information System, Researches, Rough Sets, Sheaves

1. INTRODUCTION

One can describe Rough sets by two ways. One is constructive approach and the other is algebraic approach. Upper and lower approximations are developed in constructive approach from various concepts of binary relations, partitions, coverings defined on the universe. Partially ordered sets, lattices, Boolean algebras and the sub-algebras related to it are also explored by Pawlak (1982) and Milan Vlach (2008).

Algebraic approaches to theory of Rough sets are developed by considering upper and lower approximations as fundamental notions. Yao (2003) analysed various rough set models and related algebras. A set of axioms on approximation operators are discussed by Zakowski (1983). Connections between rough sets and information systems was studied by S.D. Comer (1993), the analysis is made within the framework of Pawlak (1982) information systems. Propositional modal logic and rough set theory can be related to Boolean algebra with added operators discussed by Y.Y. Yao and T.Y. Lin (1996). E.K.R. Nagarajan et al. (2013) gave a characterization and studied algebraic properties of rough sets based on quasi-order.

Jean Leray introduced Sheaf representations. To deduce global properties from the local data, Sheaves constructed over topological spaces are suitable. Comer (1971), Keimel (1971), Hofmann (1972), Davey (1973), Swamy (1974) studied about the representations of algebraic structures through sheaves defined over topological spaces. Swamy (1974) and Wolf (1974) independently developed the sheaf construction mechanism based on Chinese Remainder Theorem for a given universal algebra. Using equivalence relations (congruences) on the set (algebra), Swamy (1974) developed the sheaf constructions.

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The applications of sheaves to various algebraic systems and real-world problems are explored consequently by researchers. J. Lambek et al. (1982) drawn analogies between Topos theory and ring theory and studied the sheaf representations on Topos. Leuştean (2005) studied the sheaf representations on BL-algebras. Y.V. Srinivas (1993) applied the sheaf theoretic approach to pattern matching and explored the new mechanisms. Ghrist et al. (2011) developed models that applies sheaf representations to Network coding.

In recent times Kishore et al. (2016) illustrated a mechanism to construct sheaves based on tolerance relations. Padmini Srinivasan (1989), H. Jiang (2009) proposed an Information retrieval model based on rough set theory and demonstrated how queries can be answered with more relevant information. Rama Murthy Garimella et al. (2015) proposed a rough set theory-based mechanism for audio, video object retrieval from multimedia databases. Butler et al. (2012) proposed a sheaf data model in which data is represented in the form of tables similar to that of relational data base model, but the rows in the tables are equipped with a user defined lattice order. In the literature it is observed that methods are proposed for information retrieval based on rough sets, data representations based on sheaf theoretic models. However, no work could be identified that combines all the three aspects to best of the knowledge of authors. In the present work the connections between sheaves, rough sets and the information systems is explored. In particular a sheaf of sets is constructed over an information system in which each stalk represents an approximation space. The advantages of such representations are analysed in information retrieval.

2.PRELIMINARIES

We presented some fundamental concepts of sheaves and some of the definitions in this section.

Definition 2.1. A relation defined on a set is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Definition 2.2. Consider a non-empty finite set B and equivalence θ on B. The set of all equivalences on B is represented by $E(B)$. For any $a \in B,$ an equivalence class of an element c is $\theta(c) = \{b \in B \mid (c,\ b) \in \theta\},$ we denote $\{\theta(c) \mid c \in B\}$ by $B \mid \theta$. Two equivalence classes E and E' are either disjoint or equal.

Definition 2.3. Let R be an equivalence relation defined on the universe V. For any subset $X \subseteq V$, the approximation space is denoted by the pair $S = (V, R)$.

The subsets $\overline{R}X = \{x \in V \mid [x]_R \cap X \neq \phi\}$, $\underline{R}X = \{x \in V \mid [x]_R \subseteq X\}$ are called the R-upper and R-lower approximation of X, respectively. Where $[x]_R$ denotes the equivalence class of the element x with respect to the equivalence relation R and $RX = (\underline{R}X, \overline{R}X)$ represents the rough set of X in S. The rough set RX describe X in terms of the information granules under the present knowledge, i.e., the classification of V.

Definition2.4: A relation defined on a non-empty finite set is said to be a quasi-ordering if it is reflexive and transitive.

Notation2.5: If β be a quasi-ordering on A and if $a\beta b$, for $a,\ b \in A$ we denote it by $\langle a,b\rangle \in \beta.$

Definition2.6 (E.K.R. Nagarajan)**:** Let β be a quasi-ordering defined on a finite non-empty set B. The set consisting of all quasi-orderings on B is represented by $Q(B)$. For any $c \in B,$ the quasi-ordered class of c is $[c]_\beta = \{b \in B \mid \langle c,\ b\rangle \in \beta\},$ we denote $\{[c]_\beta \mid c \in B\}$ by $B \mid \beta$.

Definition 2.7: Let β be a quasi-ordering defined on the universe V. For any subset $X \subseteq V$, the pair $S = (V,\beta)$ is called quasi-ordered approximation space. The β- upper and β-lower approximations of X are given by $\overline{\beta}X = \{x \in V | [x]_\beta \cap X \neq \phi\}$, $\underline{\beta}X = \{x \in V | [x]_\beta \subseteq X\}$, respectively. Where $[x]_\beta$ denotes the quasi-ordered class of the element *x* with respect to the quasi-ordering β and the quasi-rough set of *X* in *S* is $\beta X = (\underline{\beta}X, \overline{\beta}X)$. The rough set βX describes *X* under the quasi ordering β.

Definition 2.8: Let *X* be a non-empty set and T be a collection of subsets of *X* satisfying the following three conditions:

1. T shall contain empty set and the total set X
2. T is closed under finite intersections
3. T is closed under arbitrary unions

Then T is called a topology on *X*, the members of T are called open sets and the pair (X,T) is called a topological space.

A set *X* on which a topology T has been specified is called a topological space (X, T). In general, we say X is a topological space.

Example: Let $X = \{1, 2, 3, 4\}$ and $T = \{\phi, X, \{1,2,3\}, \{2, 3\}, \{2, 3, 4\}\}$ be some collection of subsets of *X*.

Clearly ϕ and *X* are in T. Also, T is closed under arbitrary unions and finite intersections.

Hence, we can say that *X* is a topological space.

Definition 2.9: A subset U of X is said to be open in X if for each $y \in U$ there is a basis element $B \in \beta$ such that $y \in B$ and $B \subseteq U$.

Definition 2.10: Let (X,τ) and $(Y,\rangle)$ be topological spaces. A mapping $f : X \to Y$ is said to be continuous if it brings back open sets in Y to open sets in X. That is for each W open in Y $f^{-1}(\mathrm{W})$ is open in X.

Definition 2.11: Let (X,τ) and $(Y,\rangle)$ be topological spaces and let f be a mapping of X into Y. Then f is said to be a homeomorphism if and only if f is isomorphic, continuous and an open map between the two topological spaces.

Definition 2.12: Let (X,τ) and $(Y,\rangle)$ be topological spaces and let f be a mapping of X into Y is a local homeomorphism if for every point $x \in X$ there exists open sets G, D containing X, $f(X)$, respectively such that the restriction $f|_G : G \to D$ is a homeomorphism.

Definition 2.13: Let M and N be sets. The disjoint union of M and N is the set represented by $M \sqcup N$ of elements of the form (a, M), (b,N) where $a \in M$ and $b \in N$, symbolically we write $M \sqcup N = \{(a,M) | a \in M\} \cup \{(b,N) | b \in N\}$

Example: Let $M = \{1,2\}$ and $N = \{2,3\}$. The disjoint union of M and N is $M \sqcup N = \{(1,M),(2,M),(2,N),(3,N)\}$

A sheaf is a mathematical tool for storing locally defined information attached to the open sets of a topological space.

Definition 2.14 (Swamy, 1974)**:**

A sheaf is represented by a triple $\left(G,\ \pi,\ Y\right)$ which satisfies the conditions:

a. *G* and *Y* be two topological spaces.
b. $\pi : G \rightarrow Y$ be the local homeomorphism from *G* onto *Y*.

That is, $\pi : G \rightarrow Y$ is a surjection in such a way that for any $g \in G$, there exists open sets A, B in $G,\ Y$, respectively, such that $g \in A,\ \pi\left(g\right) \in B$. Also $\pi\mid_A : A \rightarrow B$ is a homeomorphism. *G* and *Y* are named as the sheaf space and the base space respectively and π is named as the projection map. Often we called that $\left(G,\ \pi,\ Y\right)$ is a sheaf over Y. For any $p \in Y,\ \pi^{-1}\left(p\right) \neq \varnothing$ and is named the stalk at p, we represent this by G_p. Note that *G* is the disjoint union of all $G_p^{'}s$.

Definition 2.15: Let $\left(G,\ \pi,\ Y\right)$ be a sheaf and $X \subseteq Y$. A section on X is a continuous mapping $f : X \rightarrow G$ such that $\pi \circ f = \mathit{Identity}$. Sections on the total space Y are called global sections.

Definition2.16: Let $\left(G,\ \pi,\ Y\right)$ be a sheaf. If for every $g \in G$, there is a global section f and $y \in Y$ such that, $g \in f\left(y\right)$, then $\left(G,\ \pi,\ Y\right)$ is called a global sheaf.

Definition 2.17 (Burris and Sankappanavar, n.d.)**:** A Boolean algebra is an algebra with two binary operations $\vee,\ \wedge$ one unary operation of complementation and two nullary operations namely least and greatest elements and is denoted by $\left(B, \vee, \wedge, ', 0, 1\right)$ is satisfying the following:
$\left(B,\ \vee,\ \wedge\right)$ is a distributive lattice

$$y \wedge 0 \approx 0,\ y \vee 1 \approx 1$$
$$y \wedge y' \approx 0,\ y \vee y' \approx 1$$

Definition 2.18: An ideal I becomes a prime ideal of a Boolean algebra if $1 \notin I$ and $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$.

3. SHEAF OF SETS OVER AN INFORMATION SYSTEM

Consider $I = \left(V, A\right)$ be an Information system where V denotes the universe of objects which are finite and A denote the attributes set.

Let *Z* be the Boolean algebra defined on the subsets of A together with the ordering defined by set inclusion, that is, $Z = \left(P\left(A\right), \cup, \cap, ', \varnothing, A\right)$ where $P\left(A\right)$ denotes the set of all subsets of A.

Let X be the set which consists of all prime ideals of *Z*.

Theorem 3.1: Let A be any finite set and the Boolean algebra defined on the subsets of A be $\boldsymbol{Z} = \left(P\left(A\right),\ \cup,\ \cap,\ ',\ \varnothing,\ A\right)$. Then every prime ideal of *Z* is of the form $P\left(A - \left\{a\right\}\right)$ for some $a \in A$, where $P\left(A - \left\{a\right\}\right)$ is the power set of $\left(A - \left\{a\right\}\right)$

Proof: We show that first $P = P\left(A - \left\{a\right\}\right)$ is a prime ideal. Choose $A_1, A_2 \in P$. Clearly $A_1 \cup A_2 \in P, A_1 \cap A_2 \in P$. Now for any $C \in Z$ and $D \in P$ it can be observed that $a \notin C \cap D$, since $a \notin D$ therefore, $C \cap D \in P$.

Suppose for some $B,C \in Z$, let $B \cap C \in P$ which implies $a \notin B \cap C$. Then it follows that $a\, not\, in\, B$ (or) $a\, not\, in\, C$. Therefore $B \in P$ or $C \in P$ which shows that *P* is a prime ideal.

Conversely, suppose that *P* is a prime ideal. Then *P* is maximal, since every prime ideal is maximal and hence $P \neq Z$ there exists $C \in Z$ such that $C \notin P$.

Claim: $C = \{a\}$ for some a .

Suppose $C = \{a_1, a_2\} \notin P$. By above observation $P(A - \{a_1\})$, $P(A - \{a_2\})$ are prime ideals which are in turn maximal ideals. Therefore $P \subseteq P(A - \{a_1\})$ and *P* is maximal which is a contradiction.

3.2. Structure of Indiscernibility Relation

Consider $I = (V, A)$ be an Information system where V denotes the universe of objects and *A* denote the attribute set and $Z = (P(A), \cup, \cap, ', \varnothing, A)$ be a Boolean algebra. Then every prime ideal *P* of *Z* is of the form $P(A - \{a\})$ for some $a \in A$.

Let X be the set which consists of all prime ideals of *Z*.

For each $B \subseteq A$, define $N_B = \{P \in X \mid B \notin P\}$.

Consider the topology T on X for which the set $\{N_B \mid \; B \subseteq A\}$ forms a base. $(X, \; T)$ is called the spectrum of X is denoted by $Spec(X)$.

For $P \in X$, define an indiscernibility relation IND_P on $P(V)$ by

$$IND_{\mathbf{P}} = \left\{ \begin{array}{c} \langle o_1, o_2 \rangle \in P(V) \times P(V) \mid For\, each\, x \in o_1 \, \exists \; y \in o_2 \\ such\, that\, a(x) = a(y) \forall a \in B, \; B \in P \end{array} \right\}$$

Clearly, IND_P satisfies reflexive property and transitive property but need not satisfies symmetric property. Hence it is the quasi-ordering on V and (V, IND_P) is a quasi-ordered approximation space.

3.3. Construction of a Sheaf Through Quasi-ordering

Given a collection of documents, each document may share key words with several documents. Documents can be clustered in several ways. For example, clustering the documents that share common key words that share at least some key words may result in equivalence classes, tolerance classes respectively. However, in the huge collection of web-based objects this may result in large number of clusters. In such case maintaining the meta data itself becomes a huge task. In this regard in the present work it is proposed to group the documents in such a way that each document may belong to several clusters based on the presence of key words by using the following quasi ordering.

Let $I = (V, A)$ be an information system. Consider a topological space Y defined on a set of attributes. For $P \in Y$, define an indiscernibility relation IND_P on $P(V)$ by

$$IND_P = \left\{ \begin{array}{c} \langle o_1, o_2 \rangle \in P(V) \times P(V) \mid For\, each\, x \in o_1 \, \exists \; y \in o_2 such\, that \\ a(x) = a(y) \forall a \in P \end{array} \right\}$$

Clearly IND_P satisfies reflexive property and transitive property but need not satisfies symmetric property and hence IND_P belongs to $Q(A)$.

Consider $G_P = V / IND_P$ and $G = \bigcup^{+}_{P \in Y} G_P$ be the disjoint union. Define $\hat{o} : Y \to G$ as $\widehat{o}\left(P\right) = IND_P\left(o\right)$, for any $o \in V$. $IND_P\left(o\right)$ is the quasi ordered class of o. Topologise G with the largest topology with respect to each $\hat{o}$ is continuous.

Define $\pi : G \to Y$ by $\pi\left(IND_P\left(o\right)\right) = P$, $\forall\ IND_P\left(o\right) \in G_P$. Then $\left(G, \pi, Y\right)$ forms a triple.

Theorem 3.4: The triple $\left(G, \pi, Y\right)$ defined as above is a sheaf over an information system if and only if for every $o_1, o_2 \in V$, $Y\left(o_1, o_2\right) = \left\{P \in Y \mid \widehat{o_1}\left(P\right) = \widehat{o_2}\left(P\right)\right\}$ is open in Y.

Proof: Consider the global sheaf $\left(G, \pi, Y\right)$. We now prove that first for $o_1 \in V, \widehat{o_1}$ is a global section. From the definition it is clear that $\widehat{o_1}$ is continuous. Moreover $\pi \circ \widehat{o_1}\left(P\right) = \pi\left(\widehat{o_1}\left(P\right)\right) = \pi\left(\eta_P\left(o_1\right)\right) = P,$ for all $P \in Y$ which implies $\pi \circ \widehat{o_1} = Identity$ and we conclude that $\widehat{o_1}$ is global. We now prove $Y\left(o_1, o_2\right)$ is open in Y. Let $P \in Y\left(o_1, o_2\right)$ that is, $P \in Y, \widehat{o_1}\left(P\right) = \widehat{o_2}\left(P\right) = g$ (say), $g \in G$. Now there exists two open sets L and W in G and Y, respectively, from sheaf definition in such a way that $g \in L$ and $\pi \mid_L : L \to W$ is a homeomorphism. Observe that $\pi\left(g\right) = \pi\left(\widehat{o_1}\left(P\right)\right) = P,\ P \in W$. Now take $K = \widehat{o_1}^{-1}\left(L\right) \cap \widehat{o_2}^{-1}\left(L\right) \cap W.$ Since $\widehat{o_1}, \widehat{o_2}$ are continuous and W is open, implies that K is open in Y and also $P \in K$. For any $Q_1 \in K,$ $\widehat{o_1}\left(Q_1\right), \widehat{o_2}\left(Q_1\right) \in G$ and $\pi\left(\widehat{o_1}\left(Q_1\right)\right) = \pi\left(\widehat{o_2}\left(Q_1\right)\right)$ implies $\widehat{o_1}\left(Q_1\right) = \widehat{o_2}\left(Q_1\right)$, since $\pi \mid_L$ is a one-one map. Then $Q_1 \in Y\left(o_1, o_2\right)$ implies $Y\left(o_1, o_2\right)$ is open.

Conversely assume that $Y\left(o_1, o_2\right)$ is open in Y. Now we claim that $\left(G, \pi, Y\right)$ is a global sheaf. Let $g \in G.$ Then there exists $P in Y, o_1 \in V$ such that $g \in \eta_P\left(o_1\right).$ Now since $\eta_P\left(o_1\right) = \widehat{o_1}\left(P\right),\ \widehat{o_1}\left(P\right) \in \widehat{o_1}\left(Y\right)$ implies that $g \in \widehat{o_1}\left(Y\right)$.

Now we claim that $\pi \mid_{\widehat{o_1}\left(X\right)} : \widehat{o_1}\left(Y\right) \to Y$ is a homeomorphism.

Suppose, let $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_P\left(o_1\right)\right) = \pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_{Q_1}\left(o_1\right)\right).$ From the definition of π, $P = Q_1$ implies $\eta_P\left(o_1\right) = \eta_{Q_1}\left(o_1\right)$ and therefore $\pi \mid_{\widehat{o_1}\left(Y\right)}$ is one-one. Let $P \in Y,$ observe that $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_P\left(o_1\right)\right) = P$ for $o_1 \in V,\ \eta_P\left(o_1\right) \in \widehat{o_1}\left(Y\right).$ Then $\pi \mid_{\widehat{o_1}\left(Y\right)}$ is onto. Let W be the open set in Y and $g \in \left(\pi \mid_{\widehat{o_1}\left(Y\right)}\right)^{-1}\left(W\right)$. Then $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(g\right) \in W.$ Now since $g \in G_P$ for some P, there exist $o_1 \in V$ such that $g = \eta_P\left(o_1\right)$ and hence $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_p\left(o_1\right)\right) \in W.$ Since $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_P\left(o_1\right)\right) = P,$ it follows that $P \in W,$ clearly $\widehat{o_1}\left(P\right) \in \widehat{o_1}\left(W\right)$. As $\widehat{o_1}$ is an open map, implies $\widehat{o_1}\left(W\right)$ is open in $G.$

Let $g' \in \widehat{o_1}\left(W\right)$ then $g' = \widehat{o_1}\left(Q_1\right)\left(= \eta_{Q_1}\left(o_1\right)\right)$ for $Q_1 \in W$.

It can be noted that $\pi \mid_{\widehat{o_1}\left(Y\right)}\left(\eta_{Q_1}\left(o_1\right)\right) \in W$ as $\pi\left(\eta_P\left(o_1\right)\right) = Q_1.$ Therefore $g' = \eta_{Q_1}\left(o_1\right) \in \left(\pi \mid_{\widehat{o_1}}\right)^{-1}\left(W\right).$ Thus $\widehat{o_1}\left(W\right) \subseteq \pi \mid_{\widehat{o_1}\left(Y\right)}\left(W\right)$ which implies $\pi \mid_{\widehat{o_1}\left(Y\right)}$ is continuous.

Let M be the open set in $\widehat{o_1}\left(Y\right)$. Since the subspace topology induced by G, there is an open set N in G such that $M = \widehat{o_1}\left(Y\right) \cap N$. Let $g \in M$, then there is $Q_1 \in Y$ such that $g = \widehat{o_1}\left(Q_1\right)\left(= \eta_{Q_1}\left(o_1\right)\right)$, $g \in N$. Since $Q_1 \in \widehat{o_1}^{-1}\left(N\right)$, consider $\left(N\right) \cap Y$. Obviously $Q_1 \in J$ and J is open set in Y. Consider $Q_1 \in J$, that is, $P \in \widehat{o_1}^{-1}\left(N\right) \cap Y$. Then $\widehat{o_1}\left(P\right) \in N$ and since $\widehat{o_1}\left(P\right) \in \widehat{o_1}\left(Y\right)$, it implies that $\widehat{o_1}\left(P\right) \in \widehat{o_1}\left(Y\right) \cap N = M . P = \pi\,|_{\widehat{o_1}\left(Y\right)}\left(\widehat{o_1}\left(P\right)\right) \in \pi\,|_{\widehat{o_1}\left(Y\right)}\left(M\right)$. Therefore $\pi\,|_{\widehat{o_1}\left(Y\right)}$ is open.

Hence $\left(G,\ \pi,\ Y\right)$ is a sheaf over an information system. Here G_P is called a stalk which describes an approximation space on V corresponding to the attribute set $P \subseteq A$.

Note 3.5:

In the above construction we may choose a convenient topology on the set of all attributes and obtain the corresponding sheaf of sets over an information system. Similarly by considering different indiscernibility relations on the set of all objects we get suitable sheaf spaces.

4. ILLUSTRATION

Example 4.1: Consider an information system given in Table 1.

In the above information system $V = \left\{V1,\ \ V2,\ \ V3\right\}$ and $A = \left\{C,\ H,\ W,\ P\right\}$

The power set of V, $P\left(V\right) = \left\{\left\{\varnothing\right\},\left\{V1\right\},\left\{V2\right\},\left\{V3\right\},\left\{V1,V2\right\},\left\{V2,V3\right\},\left\{V1,V3\right\},\left\{V1,V2,V3\right\}\right\}$. Obviously $P\left(V\right) \times P\left(V\right)$ consists of 64 elements.

Consider the topology $X = \left\{\varnothing,\ \ \left\{C,\ H\right\},\ \left\{W,\ \mathrm{P}\right\},\ \left\{C,\ H,\ W,\ P\right\}\left(= A\right)\right\}$

Let us denote $P_1 = \varnothing$, $P_2 = \left\{C,\ H\right\}$, $P_3 = \left\{W,\ \mathrm{P}\right\}$, $P_4 = \left\{C,\ H,\ W,\ P\right\}\left(= A\right)$

By the definition of the indiscernibility relation defined in 3.2

The indiscernibility relations are given as follows.

$$IND_{P_1} = \begin{Bmatrix} \left\langle \varnothing,\varnothing \right\rangle,\left\langle \varnothing,\left\{V1\right\} \right\rangle,\left\langle \varnothing,\left\{V2\right\} \right\rangle,\left\langle \varnothing,\left\{V3\right\} \right\rangle,\left\langle \varnothing,\left\{V1,V2\right\} \right\rangle, \\ \left\langle \varnothing,\left\{V2,V3\right\} \right\rangle,\left\langle \varnothing,\left\{V1,V3\right\} \right\rangle,\left\langle \varnothing,\left\{V1,V2,V3\right\} \right\rangle \end{Bmatrix}$$

Table 1. Domains of keywords

Group of Documents (Objects) (V)	Domains of key words (Attributes) (A)			
	Chemistry (C)	History (H)	Wealth (W)	Physics (P)
$V1$	organic	king	Gold	rays
$V2$	reaction	rule	Villa	opaque
$V3$	organic	king	Gold	magnet

$$IND_{P_2} = \text{”} \cup \left\{ \begin{array}{l} \langle\{V1\},\{V3\}\rangle,\langle\{V1\},\{V1,V2\}\rangle,\langle\{V1\},\{V2,V3\}\rangle,\langle\{V1\},\{V1,V3\}\rangle, \\ \langle\{V2\},\{V2,V3\}\rangle,\langle\{V2\},\{V1,V2,V3\}\rangle,\langle\{V3\},\{V1,V3\}\rangle, \\ \langle\{V3\},\{V2,V3\}\rangle,\langle\{V3\},\{V1,V2\}\rangle,\langle\{V3\},\{V1,V2,V3\}\rangle, \\ \langle\{V1,V3\},\{V1\}\rangle,\langle\{V1,V3\},\{V3\}\rangle,\langle\{V1,V3\},\{V1,V3\}\rangle,\langle\{V1,V3\},\{V2,V3\}\rangle, \\ \langle\{V1,V3\},\{V1,V2,V3\}\rangle,\langle\{V1, V2\}, \{V1, V2\}\rangle, \langle\{V1, V2\}, \{V2, V3\}\rangle, \\ \langle\{V1, V2\}, \{V1, V2,V3\}\rangle,\langle\{V1,V2,V3\}, \{V1,V2\}\rangle, \\ \langle\{V1,2,V3\},\{V2,V3\}\rangle,\langle\{V1,V2,V3\},\{V1,V2,V3\}\rangle \end{array} \right\}$$

Where $\text{”} = \{\langle\{V1\},\{V1\}\rangle,\langle\{V2\},\{V2\}\rangle, \langle\{V3\},\{V3\}\rangle\}$

Similarly other indiscernibility relations can be described.

Observe that the quasi-ordered classes of IND_{P_2} are

$$[\{V1\}]_{IND_{P_2}} = [\{V1\},\{V3\}, \{V1, V3\}, \{V1, V2\}, \{V2, V3\}, \{V1, V2, V3\}]$$

$$[\{V2\}]_{IND_{P_2}} = [\{V2\}, \{V1, V2\}, \{V2,V3\}, \{V1, V2, V3\}]$$

$$[\{V1\}]_{IND_{P_2}} = [\{V3\}]_{IND_{P_2}}$$

$$[\{V1,V2\}]_{IND_{P_2}} = [\{V1,V2\},\{V2,V3\},\{V1,V2,V3\}]$$

$$[\{V1,V3\}]_{IND_{P_2}} = [\{V1\},\{V3\},\{V1,V3\},\{V2,V3\},\{V1,V2,V3\}]$$

$$[\{V2,V3\}]_{IND_{P_2}} = [\{V1\},\{V2,V3\},\{V1,V2,V3\}]$$

$$[\{V1,V2,V3\}]_{IND_{P_2}} = [\{V1,V2\},\{V2,V3\},\{V1,V2,V3\}]$$

Observe that $G_{P_2} = V / IND_{P_2}$ represents a stalk with the above quasi-ordered classes and (V, IND_{P_2}) is a quasi-ordered approximation space.

$G = G_{P_1} \sqcup G_{P_2} \sqcup G_{P_3} \sqcup G_{P_4}$ (disjoint union) that is, $\bigcup_{p_j X} \{(g, p_j) : g \in G_{P_i}\}$ and by the above sheaf construction given in 3.3 (G, π, Y) is a sheaf where $\pi : G \to Y$ is given by $\pi([o]_{IND_{P_2}}) = P_2$ where $o \subseteq V$.

5. APPLICATION OF SHEAF REPRESENTATION TO INFORMATION RETRIEVAL

Consider a set of documents (V). Let A denote the domains of key words. Let $F : V \to A$ be a map that assigns for each document the key words present in it. Then (V, F, A) denotes an information system. We can construct a sheaf over a prescribed topology for this given information system as illustrated above. In the sheaf every stalk contains quasi-ordered classes for a prescribed set of key words.

The storage of the metadata in the above format for the given information system facilitates quick query processing on the corresponding stalk. Given a query with a set of key words the corresponding smallest element P in the topology is considered and the corresponding information from $V \,/\, IND_{\mathrm{P}}$ is retrieved from the relevant stalk.

In the above example 4.1, if “organic” and “king” are selected as keywords, then it corresponds to the set $P_2 = \left\{C,\ \ H\right\}$ and the stalk information corresponding to P_2 indicates that $\left[\left\{V1\right\}\right]_{IND_{P_2}} = \left[\left\{V3\right\}\right]_{IND_{P_2}}$ and hence documents $V1, V3$ are the relevant documents and will be given as an preliminary output. Further, the quasi ordering classes corresponding to IND_{P_2} can also be presented but with lesser priority.

Similar schema and construction can also be applied to any scenario where suitable relations can be built. For example, in case of networks, nodes can be considered as objects and characteristics of nodes or other nodes connected to this node can be considered as attributes.

6. CONCLUSION

In this paper a new quasi ordering is proposed on the collection of documents based on key words which facilitates grouping of documents can be carried out in a more general and meaningful setting, and also a sheaf theoretic modal for information storage and retrieval using quasi-ordered approximation spaces is presented and demonstrated how the method can be used in real time systems. Further this can be implemented and tested on real time data to observe the improvements in quality of relevant answers and time taken for query processing.

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